# Combinatorial and Algorithmic Game Theory

## Doctural Study Report Defense

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# Combinatorial games

- Sequential games
- Perfect information
- No randomness

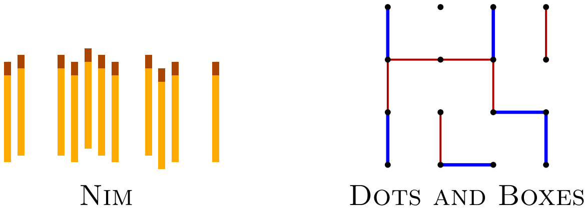



Figure: Example positions in classical combinatorial games

# Win condition for combinatorial games

| Play type | Win condition | Loose condition | Tie condition |
|---|---|---|---|
| normal play | last to move | has no move | otherwise |
| misère play | has no move | last to move | otherwise |

Table: Play types and their win and loose conditions

| Game | First player's win condition | Second player's win condition | Tie condition |
|---|---|---|---|
| Tic-Tac-Toe | mark a row with X's | mark a row with O's | otherwise |
| Maker-Breaker | color a set | color a hitting set | cannot happen |
| Online Ramsey | force creation of a monochromatic subgraph | otherwise | cannot happen |
| Eternal domination | defend the graph ad infinitum | otherwise | cannot happen |

Table: Games with their win condition for each player

# Solving games

| | |
|---|---|
| Ultra-weak | we only know the outcome, no strategy |
| Weak | outcome and strategy is shown for the initial position |
| Strong | we can compute the optimal move for any position |

# Work so far

- Swap pattern matching (unrelated)
  - Published in MACIS 2017
- Online Ramsey theory
  - Accepted to CSR 2019
- m-Eternal domination problem
  - Submitted to RC 2019
- Edge-length ratio of $2$-trees (weakly related)
  - To be submitted

# Online Ramsey number - Game rules

The *online Ramsey game*:

- Two players, Builder and Painter, play against each other.
- They are given a graph $H$ and a playfield of an infinite set of independent vertices.
- Each round Builder draws an edge and Painter colors it either **red** or **blue**.

Goals of the players are:

- Builder's goal is to create a monochromatic target graph $H$.
- Painter's goal is to not lose for as many rounds as possible.

# Online Ramsey number

## Definition

The *online Ramsey number* is the minimum number of rounds such that Builder has a winning strategy in the online Ramsey game if both players play optimally.

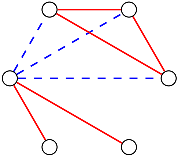
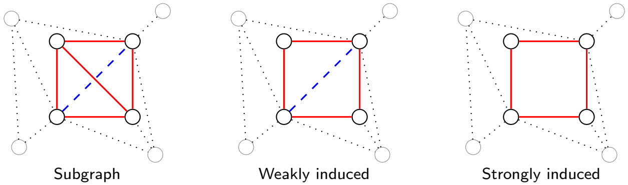


Figure: ORN$(C_3) \leq 8$.

# Induced online Ramsey number




$$\text{ORN}(H) \leq \text{ORN}_{wind}(H) \leq \text{ORN}_{ind}(H)$$

Strongly induced constructive results:

- paths in $O(n)$
- cycles in $O(n)$
- $l$-subdivided stars $S_k$ in $O(k^2 l)$
- $l$ stars $S_k$ on a common path in $O(kl)$

Joint work with Tomáš Valla and Pavel Dvořák.

# m-Eternal domination problem - Game rules

The m-Eternal domination game:

- Two players play against each other.
- They are given a graph as a playfield.
- The first player controls a set of vertex guards.
- The second player chooses a vertex to attack each turn.
- Each turn the first player wants to move his guards such that there is a guard on the attacked vertex.

Goals of the players are:

- The first player wants to defend against the attacks forever.
- The second player wants to find a sequence of attacks which is udefendable.

# m-Eternal domination problem

## Definition

The *m-eternal domination number* of a graph is the lowest number of guards such that the first player wins.

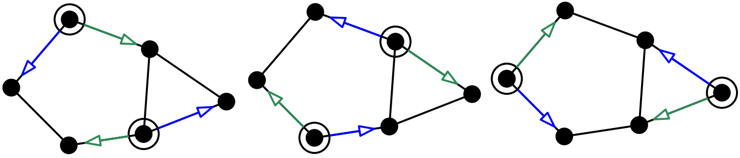


Figure: Example eternal dominating strategy on a graph using $2$ guards; Same colored arrows show how the guards would move at once.

# m-Eternal domination problem

- upper bound for cactus graphs
- solution for christmas cactus graphs

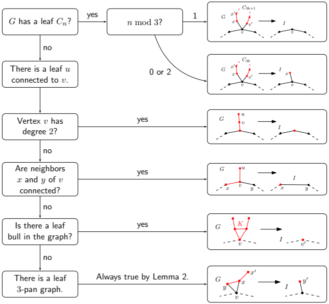
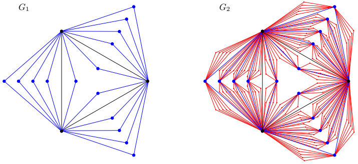



Figure: Descision tree for chosing a proper reduction

Joint work with Jan Matyáš Křišťan and Tomáš Valla.

# Edge-length ratio of $2$-trees (slightly off topic)

**Definition**
The *edge-length ratio of a graph drawing* is $\frac{L}{S}$, where $L$ is length of the longest edge and $S$ is length of the shortest edge.




**Definition**
The *local edge-length ratio of a graph drawing* is the biggest ratio of lengths of two incident edges.

We showed that

- the edge-length ratio of $2$-trees is unbounded,
- the local edge-length ratio of $2$-trees has upper bound $(4 + \epsilon)$.

Joint work with Jiří Fiala and Giuseppe Liotta.

# Summary of the results

- Online Ramsey theory
  - induced paths in $O(n)$
  - induced cycles in $O(n)$
  - induced $l$-subdivided stars $S_k$ in $O(k^2 l)$
  - induced $l$ stars $S_k$ on a common path in $O(kl)$
- m-Eternal domination problem
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- Edge-length ratio of $2$-trees
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(some other notions and future work is next)

# Complexity of combinatorial games

In general, combinatorial games are quite hard, game with $x$ rounds can be thought of as searching for solution in

$$(\exists A_1)(\forall B_1)(\exists A_2)(\forall B_2)\dots(\exists A_x) \text{ such that there is no move.}$$

It was noted by Fraenkel et al. that the game tends to be PSPACE-complete if it has bounded number of quantifiers, and if it is unbounded it is usually EXPTIME-complete.

- The complexity classes for combinatorial games were proposed but are not widely used.
- These classes seem to have no complete problems yet.

# Algorithmic game theory

- Perfect information
- Playing simultaneously
- Each agent (player) maximizes his payoff

A choice where all agents cannot change their choice to make their payoff better, is called *Nash equilibrium*.

We can use *mechanism design* to tackle some problems.

- Design payoffs in a way such that all the Nash equilibria yield a solution to some problem.
- Prove that a Nash equilibrium can be found quickly.

# Proposed doctoral thesis

Title of the thesis: *Combinatorial and Algorithmic Game Theory*

We suggest to focus on the following:

- Find solutions to combinatorial games,
- investigate their complexity and gather evidence of notions which might be used to categorize combinatorial games into (potentially new) complexity classes,
- design algorithmic games where the result corresponds to a globally desirable outcome, mainly (but not only) for combinatorial optimization problems,
- investigate its algorithmic and computational complexity aspects, investigate common phenomena of the original problems and their game theoretical formulations.